Sequences and Series- Ms. Pilch's problems

1. Find:
   a) $\sum_{k=1}^{8} 3k + 1$
   b) $\sum_{i=0}^{5} \frac{1}{i!}$

2. Find the 3rd partial sum of the following series: $\sum_{i=1}^{\infty} \frac{5}{10^i}$

3. Find the sum of the following infinite series: $\sum_{i=1}^{\infty} \frac{5}{10^i}$

4. Find the sum of the following infinite series: $\sum_{i=0}^{\infty} \frac{1}{i!}$

Express the given series using sigma notation:

5. $5 + 9 + 13 + ... + 101$

6. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$

7. $7 + 3 - 1 - 5 - 9 - 13$

8. $3\sqrt{2} + 7\sqrt{2} + 11\sqrt{2},...$
Are the two expressions equivalent? If so, put the = sign between them, and if not put the ≠ sign between them:

9. \( \sum_{i=1}^{n} (a_i + b_i) \) \( \neq \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)

10. \( \sum_{i=1}^{n} (a_i \cdot b_i) \) \( \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \)

11. \( \sum_{k=1}^{n} (-1)^k \cdot \log k \) \( \neq \log \left( \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} \right) \)

12. \( \sum_{i=1}^{n} (c \cdot a_i) \) \( \neq c \cdot \sum_{i=1}^{n} a_i \)

Write the next two apparent terms in each sequence, figure out the formula for \( a_n \) calculate \( a_{100} \).

13. \( \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots \)

14. 1, -1, 1, -1, 1, -1, ...

15. \( \frac{0}{8}, \frac{13}{27}, \frac{26}{64}, \frac{39}{125}, \frac{52}{216}, \ldots \)

16. 1, 2, 6, 24, 120, 720, ...

17. Prove that 0.9 = 1. *Hint: 0.9 = 0.9 + 0.09 + 0.009 + ....

18. Find the sum of \( 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots \) and find the **interval of convergence** for \( x \) (the set of values for \( x \) such that the series has a finite sum).
19. Find the 160th term of the sequence 6, 10, 14, 18, ....

20. Find the 47th term of the sequence, 2, 5, 10, 17, 26, ....

21. Find the sum of the first 40 terms of the arithmetic sequence, 10, 8, 6, 4, ... Express it using sigma notation as well.

22. Which term of the sequence 5, 14, 23, ... is 239?

23. Compute the sum of the first 100 integers exactly divisible by 7.

24. How many consecutive integers beginning with 10 must be taken for their sum to equal 2035?

25. Show that if the sides of a right triangle are in an arithmetic sequence, their ratio is 3:4:5.

26. How long will it take to pay off a debt of $880 if $25 is paid the first month, $27 the second month, $29 the third month, etc.? 
27. Find three numbers in an arithmetic sequence such that the sum of the first and third number is 12 and the product of the first and second is 24.

28. Three numbers are in the ratio of 2:5:7. If 7 is subtracted from the second, the resulting numbers form an arithmetic sequence. Determine the original numbers.

29. Find three geometric means between 3 and 48.

30. Find four arithmetic means between 37 and 54.

31. Insert between 1 and 36 a certain amount of arithmetic means so that the sum of the resulting arithmetic sequence will be 148.

32. Show that if \( a_1, a_2, a_3 \) are the first terms of an arithmetic progression, then \( a_2 = \frac{a_1 + a_3}{2} \)

33. Prove that if \( a_1, a_2, a_3, a_4 \) are the first four terms of an arithmetic progression, then \( \frac{2a_1 + a_4}{3} = a_2 \)
34. Find x if 2x - 1, 5x - 3, 4x + 3 is arithmetic.

35. Are the following sequences arithmetic? Support/prove your answer.
   a) b - 4k, b - 3k, b - 2k
   b) (x + y)^2, x^2 + y^2, (x - y)^2

36. Find x if the arithmetic mean of (4x + 1) and (x - 9) is (x + 2).

37. If the fourth term of an arithmetic sequence is -5 and the 10th term is 22, find d and the first term.

38. The sum of the first n positive integers is 4950. Find n.

39. A number of the form \( T_n = 1 + 2 + 3 + 4 + \ldots + n \) is called a triangular number. [For example 6 is the third triangular number since 1 + 2 + 3 = 6.] Show that for any positive integer n:
   \[ T_n + T_{n-1} = n^2. \]

40. Find the sum of the first n positive integers.
41. Find the seventeenth term in the arithmetic sequence: $3\sqrt{2}, 7\sqrt{2}, 11\sqrt{2},...$

42. Find the 31st term of the geometric sequence whose first term is 1000 and whose common ratio is .95.

43. What is the number of the term whose value is 1536 in a geometric sequence with $a_1 = 3$ and $r = 2$?

44. Give one example of an (infinite) geometric series that converges and one example of a geometric series that does not converge.

45. Assume that whenever you wash a pair of blue jeans, they lose 4% of the color they had just before the wash. Let $f(n)$ represent the percentage of color left in your jeans after $n$ washes.
   a. What is the percentage of color left in your jeans after 10 washes?

   b. After how many washes will the color left in the jeans drop below 25%?

   c. Sketch a reasonable graph of $y = f(n)$
46. Find the sum of the following infinite geometric series \( \sqrt{27} + \sqrt{9} + \sqrt{3} + 1 + \ldots \)

47. What is the value of \( x \) if the series \( x^2 - x^3 + x^4 - \ldots \) converges (sums) to \( \frac{x}{5} \)?

48. Find the interval of convergence and the sum (expressed in terms of \( x \)) of the following series.
   a) \( 1 + x^2 + x^4 + x^6 + \ldots \)
   b) \( 1 + 3x + 9x^2 + 27x^3 + \ldots \)
   c) \( 1 + (x - 3) + (x - 3)^2 + (x - 3)^3 + \ldots \)

49. Each side of an equilateral triangle has length 12. The midpoints of the sides of the triangle are joined to form another equilateral triangle, and the midpoint of this triangle are joined to form still another triangle. If this process is repeated indefinitely, find
   a. the sum of the areas of all the triangles and
   b. the sum of the perimeters.
50. Here is an old paradox: Achilles races a turtle who has a 100-meter head start. If Achilles runs 10 m/s and the turtle only 1 m/s, when will Achilles overtake the turtle?

*Erroneous Solution:* When Achilles covers the 100-meter head start, the turtle has moved 10 m ahead. And when Achilles covers this 10 m, the turtle has moved 1 m ahead. Every time Achilles runs to where the turtle was, the turtle has moved 1 m ahead. Thus, Achilles can never catch the turtle.

What is the correct solution?

51. If $2x + 1$, $3x + 2$, and $5x$ are the first three numbers in an arithmetic sequence, find $S_{20}$.

52. If $a_7 = -20$ and $a_{16} = -65$ in an arithmetic sequence, find the sum of the first 16 terms.

53. If $2x - 7$, $6x - 2$, and $8x + 4$ form an arithmetic sequence, find the value of $x$ and the term values.

54. If $5x - 3$, $x + 2$, and $3x - 11$ for an arithmetic sequence,

   a) find the value of $x$.

   b) find the $15^{th}$ term.

   c) find $S_8$
55. Find the geometric mean between $5\sqrt{2} + 1$ and $5\sqrt{2} - 1$.

56. A bookseller arranges to sell a set of books for $360 to be paid in 20 monthly installments with form an arithmetic sequence (why not?). It is also desired that all the books shall be two-thirds paid for with the payment of the tenth installment. What will be the first and second payments and the final payment?

57. The sum of three consecutive numbers in an arithmetic sequence is 52. The third number is 2.25 times the sum of the other two. Find the numbers.

58. The sum of three consecutive numbers in a geometric sequence is 52. The third number is 2.25 times the sum of the other two. Find the three numbers.

59. Three numbers who sum is 18 form an arithmetic sequence. If the first number is multiplied by 2, the second multiplied by 3 and the third multiplied by 6, the resulting numbers form a geometric sequence. Find the numbers.
60. In a square whose sides each have length 7 is inscribed in a smaller square whose vertices divide the sides of the larger square in a 4:3 ratio.

61. A third square is inscribed in the second square, again dividing the sides in a 4:3 ratio. If this process is repeated infinitely many times, what will be the sum of the perimeters of all the squares?

62. The segments joining the midpoints of the sides of an equilateral triangle are drawn, and the interior of the triangle they form is removed from the interior of the original triangle. The process is repeated as shown, with the remaining interior being the shaded region in the diagram. If this process is repeated infinitely many times, how much of the original interior will be left. Show work/proof to support your answer.
63. In an arithmetic sequence \( a_3 = 9 \) and \( a_9 = 10 \). Find
   a. \( a_1 \)
   
b. \( S_{11} \) (consider the series associated with the given sequence and its partial sums)
   
c. \( S \) (the sum of the infinite series, if it exists).

64. Note: \( n \) is a variable, so your answer may well involve \( n \). Find \( \sum_{k=1}^{n} 2k \)

65. Consider the infinite series \( 1 + \frac{3x}{2} + \frac{9x^2}{4} + ... \)
   a. For what values of \( x \) does the series converge (that is, for what \( x \) will it have a finite sum?)
   
b. If the series converge, what is its sum (expressed in terms of \( x \))?
   
c. Express the series in sigma notation.

66. Write as a single radical: \( \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[n]{a} \cdot \sqrt[5]{a} \cdot ... \sqrt[512]{a} \)
67. For what values of \( x \), if any, does the following series converge:
\[
\log_5 x + (\log_5 x)^2 + (\log_5 x)^3 + (\log_5 x)^4 + ...
\]

68. Simplify \( \frac{(n + 2)n!}{(n + 1)!} \) (Note: there should not be any factorial in the answer)

69. Three numbers form an arithmetic sequence whose sum is 27. If 2 is subtracted from the first term and 14 is added to the last term, the resulting sequence is geometric. Find the original arithmetic sequence.

70. \[ \sum_{k=5}^{19} 3(.9)^{k-1} \]

71. Find 6 geometric means between \( x \) and \( x^{17} \).

72. Is the following sequence geometric, arithmetic, or neither? Write an expression for \( a_n \) and find \( a_{10} \).

a) \( \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \frac{1}{5040}, \ldots \)

b) \( -81, 27, -9, 3, \ldots \)

c) \( \log 2, \log 2^2, \log 2^3, \log 2^4, \log 2^5, \ldots \)
73. What is the number of the term whose value is 1562500 in a geometric sequence with \( a_1 = 4 \) and \( r = 5 \)?

74. Find the sum of the infinite series \( \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \ldots \)

75. The 71st term of 30, 27, 24, 21, .... is ______

76. Kyle piles 150 toothpick in layers so that each layer has one less toothpick than the layer below. If the top layer has 3 toothpicks, how many layers are there?

77. Leaving your answer in terms of \( \ln \), what is the value of \( \sum_{j=3}^{5} \ln j \)?

78. \( \sum_{k=1}^{250} \left( \frac{1}{k+1} - \frac{1}{k} \right) = \)